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The Size of $\{x: r_n'/r_n \geqslant 1\}$ and Lower Bounds for $\|e^{-x} - r_n\|$

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INTRODUCTION

We derive inequalities of the form

$$\mu \left\{ x: \frac{r_n'(x)}{r_n(x)} \geqslant \alpha \right\} \leqslant \frac{\beta \cdot n}{\alpha} \qquad \text{for } \alpha > 0,$$

where r_n is a rational function of degree n, β is a constant independent of n and μ is Lebesgue measure. We then use these inequalities to construct lower bounds for the error in approximating e^{-x} on $[0, \infty)$ uniformly by rational functions.

Let Π_n denote the set of polynomials with real coefficients of degree at most n. Let Π_n^+ denote the subset of Π_n whose elements have non-negative coefficients and let $\Pi_n^{\uparrow}$ denote the subset of Π_n whose elements are non-negative and non-decreasing on $[0, \infty)$. The prototype result proved by Loomis [3] is

THEOREM A. *If $p_n \in \Pi_n$ has only real roots, then*

$$\mu \left\{ x: \frac{p_n'(x)}{p_n(x)} \geqslant \alpha \right\} = \frac{n}{\alpha} \qquad \text{for} \quad \alpha > 0.$$

We extend this result to unrestricted polynomials and various classes of rational functions. As an application we prove

THEOREM 1. *Let $\delta > 0$.*

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(a) *There does not exist a sequence* $\{p_n/q_n\}$ *where* $p_n \in \Pi_n^{\uparrow}$ *and* $q_n \in \Pi_n^+$ so that

$$\|e^{-x} - p_n/q_n\|_{[0,n+2]}^{1/n} \leqslant \frac{1}{e^{1+\delta}} \qquad \textit{for all } n.$$

(b) *There does not exist a sequence* $\{p_n/q_n\}$ *where* $p_n \in \Pi_n^{\uparrow}$ *and* $q_n \in \Pi_n$ *so that*

$$\|e^{-x} - p_n/q_n\|_{[0,2(n+2)]}^{1/n} \leqslant \frac{1}{e^{2+\delta}} \qquad \textit{for all } n.$$

(c) *There does not exist a sequence* $\{p_n/q_n\}$ *where* $p_n \in \Pi_n$ *and* $q_n \in \Pi_n$ *so that*

$$\|e^{-x} - p_n/q_n\|_{[0,8(n+2)]}^{1/n} \leqslant \frac{1}{e^{8+\delta}} \qquad \textit{for all } n.$$

If the correct order for unrestricted rational approximation to e^{-x} on $[0, \infty)$ is $1/9^n$, as is suggested by the numerical data in [2], then (b) would show that demanding the numerator be monotonic must hinder the rate of convergence. Since the order of approximation to e^{-x} on $[0, \infty)$ by reciprocals of polynomials behaves like $1/3^n$ (see [4]), part (a) shows that requiring the numerator to be non-decreasing and the denominator to have positive coefficients makes this type of rational approximation essentially slower then reciprocal polynomial approximation. We note that the constant in (c) is not as good as the lower bound of $1/54$ obtained by Blatt and Braess in [1].

INEQUALITIES

We prove the following:

Inequality 1. If $p_n \in \Pi_n$, *then*

$$\mu\left\{x: \frac{p_n'(x)}{p_n(x)} \geqslant \alpha\right\} \leqslant \frac{2n}{\alpha} \qquad \text{for } \alpha > 0.$$

There exists $p_N \in \Pi_{16}$ *so that*

$$\mu\left\{x: \frac{p_N'(x)}{p_N(x)} \geqslant 1\right\} \geqslant 1.52N.*$$

* G. K. Kristiansen (Siam Review Problem 80–16) has shown that there Exist $p_n \in \Pi_n$ so that

$$\lim_{n\to\infty} \mu\left\{x: \frac{p_n'(x)}{p_n(x)} \geqslant 2n\right\} = 1.$$

Inequality 2. (a) *If $r_n = p_n/q_n$, where $p_n, q_n \in \Pi_n$, then*

$$\mu\left\{x: \frac{r_n'(x)}{r_n(x)} \geqslant \alpha\right\} \leqslant \frac{8n}{\alpha} \qquad \text{for} \quad \alpha > 0.$$

(b) *If $r_n = p_n/q_n$, where $p_n, q_n \in \Pi_n$ and both p_n and q_n have only real roots, then*

$$\mu\left\{x: \frac{r_n'(x)}{r_n(x)} \geqslant \alpha\right\} \leqslant \frac{4n}{\alpha} \qquad \text{for} \quad \alpha > 0.$$

Let $r_n(x) = x^n/(4n - x)^n$. Then $\mu\{x: r_n'(x)/r_n(x) \geqslant 1\} = 4n$.

(c) *If $r_n = p_n/q_n$, where $p_n \in \Pi_n$ and $q_n \in \Pi_n^{\uparrow}$, then*

$$\mu\left\{x \geqslant 0: \frac{r_n'(x)}{r_n(x)} \geqslant \alpha\right\} \leqslant \frac{2n}{\alpha} \qquad \text{for} \quad \alpha > 0.$$

(d) *If $r_n = p_n/q_n$, where $p_n \in \Pi_n^+$ and $q_n \in \Pi_n^{\uparrow}$, then*

$$\mu\left\{x \geqslant 0: \frac{r_n'(x)}{r_n(x)} \geqslant \alpha\right\} \leqslant \frac{n}{\alpha} \qquad \text{for} \quad \alpha > 0.$$

Let $r_n = x^n$. Then

$$\mu\left\{x \geqslant 0: \frac{r_n'(x)}{r_n(x)} \geqslant \alpha\right\} = \frac{n}{\alpha} \qquad \text{for} \quad \alpha > 0.$$

Inequality 3. *If $p_n \in \Pi_n$ has n real roots lying in the interval (a, b), then*

$$\mu\left\{x: \left|\frac{p_n'(x)}{p_n(x)}\right| \leqslant \frac{\alpha}{|(b-x)(x-a)|}\right\} = \frac{2n}{\alpha} \qquad \text{for} \quad \alpha > 0.$$

We need the following lemma due to Videnskii [5]:

LEMMA A. (a) *Suppose $p_{2n} \in \Pi_{2n} - \Pi_{2n-1}$ and suppose that $p_{2n} \geqslant 0$ on (a, b). Then*

$$p_{2n}(x) = (x-a)(b-x)\, t_{(n-1)}^2(x) + s_n^2(x),$$

where $t_{n-1} \in \Pi_{n-1}$, $s_n \in \Pi_n$ and both t_{n-1} and s_n have only real roots.

(b) *Suppose $p_{2n+1} \in \Pi_{2n+1} - \Pi_{2n}$ and suppose that $p_{2n+1} > 0$ on (a, b). Then*

$$p_{2n+1}(x) = (b-x)\, t_n^2(x) + (x-a)\, s_n^2(x)$$

where $t_n, s_n \in \Pi_n$ and both t_n and s_n have only real roots.

Proof of Inequality 1. Let $\alpha > 0$ and let $p_n \in \Pi_n$. Choose a and b so that

$$\left\{x: \frac{p_n'(x)}{p_n(x)} \geqslant \alpha\right\} \subset [a, b].$$

By Lemma A we can find $s \in \Pi_{2n}$, $t \in \Pi_{2n}$ so that

$$p_n^2(x) = s(x) + t(x),$$

where, for $x \in [a, b]$,

$$0 \leqslant s(x) \leqslant p_n^2(x), \qquad 0 \leqslant t(x) \leqslant p_n^2(x)$$

and both s and t have only real roots.

Now

$$\left\{x: \frac{p_n'(x)}{p_n(x)} \geqslant \alpha\right\} = \left\{x: \frac{(p_n^2(x))'}{p_n^2(x)} \geqslant 2\alpha\right\}.$$

Also,

$$(p_n^2(x))' \geqslant 2\alpha p_n^2(x)$$

exactly when

$$s'(x) + t'(x) \geqslant 2\alpha(s(x) + t(x)).$$

By Theorem A,

$$\mu \left\{x: \frac{s'(x)}{s(x)} \geqslant 2\alpha\right\} = \mu \left\{x: \frac{t'(x)}{t(x)} \geqslant 2\alpha\right\} \leqslant \frac{n}{\alpha}.$$

Since s and t are non-negative on $[a, b]$, it follows that

$$\mu\{x \in [a, b]: s'(x) + t'(x) \geqslant 2\alpha(s(x) + t(x))\} \leqslant \frac{2n}{\alpha}$$

and the bound is established.

To construct a lower bound for the inequality we observe that if $0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n/2}$ and if $p_n \in \Pi_n$ is the unique polynomial satisfying

$$\begin{aligned} p_n'(x) - p_n(x) &= -x(x - a_1)^2 (x - a_2)^2 \cdots (x - a_{n/2-1})^2 (x - a_{n/2}) \\ &= -\sum_{i=0}^{n} b_i x^i, \end{aligned}$$

then

$$\frac{p_n'(x)}{p_n(x)} \geqslant 1 \qquad \text{on} \quad [0, a_{n/2}]$$

provided that $p_n(0) > 0$. since $p_n(0) = \sum_{i=0}^{n} i!\, b_i$ we have an easy criteria to check for a given choice of a_i. It is a matter of calculation that if

$$a_1 = 0.5, \quad a_2 = 1.5, \quad a_3 = 3, \quad a_4 = 5$$
$$a_5 = 8, \quad a_6 = 12, \quad a_7 = 18, \quad a_8 = 24.32,$$

then

$$\sum_{i=1}^{n} i!\, b_i \geqslant 3.64 \times 10^7.$$

Proof of Inequality 2. To prove part (a) we note that if $r_n = p_n/q_n$, then

$$\frac{r_n'}{r_n} = \frac{p_n'}{p_n} - \frac{q_n'}{q_n} \tag{1}$$

and

$$\left\{x: \frac{r_n'}{r_n} \geqslant \alpha\right\} \subset \left\{x: \frac{p_n'}{p_n} \geqslant \frac{\alpha}{2}\right\} \cup \left\{x: \frac{q_n'}{q_n} \leqslant -\frac{\alpha}{2}\right\}.$$

By Inequality 1,

$$\mu\left\{x: \frac{p_n'}{p_n} \geqslant \frac{\alpha}{2}\right\} \leqslant \frac{4n}{\alpha} \tag{2}$$

and

$$\mu\left\{x: \frac{q_n'}{q_n} \leqslant -\frac{\alpha}{2}\right\} = \mu\left\{x: \frac{q_n'(-x)}{q_n(-x)} \geqslant \frac{\alpha}{2}\right\} \leqslant \frac{4n}{\alpha} \tag{3}$$

and it follows that

$$\mu\left\{x: \frac{r_n'}{r_n} \geqslant \alpha\right\} \leqslant \frac{8n}{\alpha}.$$

To prove part (b) we observe that we can apply Theorem A instead of Inequality 1 to (2) and (3) above to obtain

$$\mu\left\{x: \frac{p_n'}{p_n} \geqslant \frac{\alpha}{2}\right\} \leqslant \frac{2n}{\alpha} \quad \text{and} \quad \mu\left\{x: \frac{q_n'}{q_n} \leqslant \frac{-\alpha}{2}\right\} \leqslant \frac{2n}{\alpha}$$

and the result proceeds analogously.

To prove part (c) we note that $q_n'/q_n \geqslant 0$ on $[0, \infty)$ and, hence,

$$\mu\left\{x \geqslant 0: \frac{r_n'}{r_n} \geqslant \alpha\right\} \leqslant \mu\left\{x: \frac{p_n'}{p_n} \geqslant \alpha\right\} \leqslant \frac{2n}{\alpha}.$$

To prove part (d) we need only note that if $p_n \in \Pi_n^+$, then $p_n'(x) \leqslant (n/x)\, p_n(x)$ for $x > 0$ and, hence,

$$\mu \left\{ x \geqslant 0 : \frac{r_n'}{r_n} \geqslant \alpha \right\} \leqslant \mu \left\{ x \geqslant 0 : \frac{p_n'}{p_n} \geqslant \alpha \right\} \leqslant \frac{n}{\alpha}.$$

The method of proof for Inequality 3 illustrates the method Loomis employed to prove Theorem 1.

Proof of Inequality 3. We will prove that

$$\mu \left\{ x : 0 \leqslant \frac{(x-a)(b-x)\, p_n'(x)}{p_n(x)} \leqslant \alpha \right\} = \frac{\alpha}{n} \tag{4}$$

and

$$\mu \left\{ x : 0 \geqslant \frac{(x-a)(b-x)\, p_n'(x)}{p_n(x)} \geqslant -\alpha \right\} = \frac{\alpha}{n}. \tag{5}$$

Let $y_0 \leqslant \cdots \leqslant y_n$ denote the $n+1$ roots of $(x-a)(b-x)\; p_n'(x)$ and let $x_0 \leqslant \cdots \leqslant x_{n-1}$ denote the n roots of $p_n(x)$. Then $y_0 \leqslant x_0 \leqslant y_1 \leqslant \cdots \leqslant x_{n-1} \leqslant y_n \leqslant x_n = \infty$. Since

$$(x-a)(b-x)\, p_n'(x)/p_n(x) \to -\infty \qquad \text{as} \quad x \to x_i \text{ from below,}$$

we deduce that on each interval (y_i, x_i) there exists a point δ_i so that

$$(\delta_i - a)(b - \delta_i)\, p_n'(\delta_i) = -\alpha p_n(\delta_i).$$

Since the above equation can have at most $n+1$ solutions, we have

$$\mu \left\{ x : 0 \geqslant \frac{(x-a)(b-x)\, p_n'(x)}{p_n(x)} \geqslant -\alpha \right\} = \sum_{i=0}^{n} (\delta_i - y_i).$$

If

$$p_n(x) = x^n + cx^{n-1} + \cdots,$$

then

$$(x-a)(b-x)\, p_n'(x) = -nx^{n+1} + [c(1-n) + n(b+a)]\, x^n + \cdots$$

and

$$(x-a)(b-x)\, p_n'(x) + \alpha p_n(x) = -nx^{n+1} + [c(1-n) + n(b+a) + \alpha]\, x^n + \cdots.$$

From this we deduce that

$$\sum_{i=0}^{n} \delta_i - y_i = \frac{\alpha}{n}.$$

Equality (4) is proved analogously.

LOWER BOUND ESTIMATES

All three parts of Theorem 1 follow from Inequality 2 and the next lemma.

LEMMA 1. *Let* $n \geqslant 100$ *and* $2n \geqslant A \geqslant \frac{1}{2}$. *If* $r_n = p_n/q_n$, $p_n q_n \in \Pi_n$, *satisfies*

$$\mu \left\{ x \geqslant 0: \frac{r_n'(x)}{r_n(x)} \geqslant \alpha \right\} \leqslant \frac{An}{\alpha},$$

then

$$\left\| e^{-x} - \frac{q_n(x)}{p_n(x)} \right\|_{[0, A(n+2)]} \geqslant \frac{1}{n^3 e^{A(n+2)}}. \tag{6}$$

Proof. Suppose (6) is false. Then for $x \in [0, A(n+2)]$,

$$\left| e^x - \frac{p_n(x)}{q_n(x)} \right| \leqslant \frac{1}{n^3} \left(\frac{n^3}{n^3 - 1} \right) e^x \leqslant \left(\frac{e^x}{n^3 - 1} \right).$$

Set $\alpha = 1 - 1/2n$, then

$$\mu \left\{ x \geqslant 0: \frac{r_n'(x)}{r_n(x)} \geqslant 1 - \frac{1}{2n} \right\} \leqslant \frac{An}{1 - 1/2n} \leqslant A(n+1).$$

The rational function r_n'/r_n is of degree at most $2n$. Thus, there exists an interval $[a, a + A/2n]$ contained in $[0, A(n+2)]$ so that

$$\frac{r_n'(x)}{r_n(x)} \leqslant 1 - \frac{1}{2n} \qquad \text{for} \quad x \in \left[a, a + \frac{A}{2n} \right].$$

Thus, for some $\zeta \in (a, a + A/2n)$

$$\begin{aligned}
2\left(\frac{1}{n^3-1}\right)e^{a+A/2n} &\geqslant \left| e^{a+A/2n} - r_n\left(a+\frac{A}{2n}\right) - e^a + r_n(a)\right| \\
&\geqslant \frac{A}{2n}\left|e^{\zeta} - r_n'(\zeta)\right| \\
&\geqslant \frac{A}{2n}\left(e^{\zeta} - \left(1-\frac{1}{2n}\right)r_n(\zeta)\right) \\
&\geqslant \frac{A}{2n}\left(e^{\zeta} - \left(1-\frac{1}{2n}\right)e^{\zeta}\left(1+\frac{1}{n^3-1}\right)\right) \\
&\geqslant \frac{A}{2n}\left(\frac{1}{2n} - \frac{1}{n^3-1}\right)e^{\zeta} \\
&\geqslant \frac{A}{8n^2}e^{\zeta} \geqslant \frac{A}{8n^2}e^{a}.
\end{aligned}$$

Equivalently,

$$\frac{16n^2}{n^3-1}e^{A/2n} \geqslant A$$

which, since $\frac{1}{2} \leqslant A \leqslant 2n$ and $n \geqslant 100$, is impossible.

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